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NONLINEAR COLLISIONLESS PERPENDICULAR DIFFUSION OF CHARGED PARTICLES

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ABSTRACT

A nonlinear theory of the perpendicular diffusion of charged particles is presented, including the influence of parallel scattering and dynamical turbulence. The theory shows encouraging agreement with numerical simulations.
Subject headings: diffusion — turbulence

1. INTRODUCTION

The behavior of charged test particles in a turbulent magnetic field is a problem of long-standing importance in space and astrophysics, and one for which it has been unexpectedly difficult to achieve closure (Jokipii 1966; Giacalone & Jokipii 1999; Mace, Matthaeus, & Bieber 2000). The diffusion tensor, for particle transport parallel and perpendicular to the ordered magnetic field, is essential for describing solar energetic particles (Droege 2000), the modulation of Galactic cosmic rays (Burger & Hattingh 1998), diffusive shock acceleration, and the lifetime of cosmic rays in the Galaxy (Jokipii & Parker 1969). Perpendicular diffusion continues to pose a number of enigmatic problems in heliospheric studies, in which observations provide strong constraints. It is also troubling that direct numerical simulations have failed to verify known theoretical formulations of perpendicular diffusion. Some problems, such as channeling or dropouts (Mazur et al. 2000) or the occurrence of subdiffusive transport (Kota & Jokipii 2000; Qin, Matthaeus, & Bieber 2002a), may require significant theoretical reformulations. In the latter case, the onset of collisionless parallel diffusion, usually the stronger of the two effects, can thwart the occurrence of perpendicular diffusion. However, perpendicular transport can be diffusive, when the three-dimensional random magnetic field possesses adequate complexity in the directions transverse to the mean field. In this Letter, we develop a nonlinear theory of perpendicular diffusion that is applicable to this incompletely explored regime.

2. APPROACH AND METHODS

The standard description of perpendicular diffusion is that the gyrocenters of charged particles follow magnetic field lines, and thus their diffusive spread perpendicular to the mean magnetic field is governed by the diffusive spread of field lines. This field line random walk (FLRW) limit of particle transport emerges from quasi-linear theory (QLT; Jokipii 1966) and provides a physically appealing picture. However, the FLRW has not proved to be accurate for all particle energies in numerical experiments (Giacalone & Jokipii 1999; Mace et al. 2000), nor is it clear that it accounts for observed cross-field transport. Test particle studies show that the FLRW is accurate at high particle energies (the Larmor radius r_L is much greater than the parallel turbulence correlation scale $\lambda_c^{\parallel}$). For medium-energy ($r_L \approx \lambda_c^{\parallel}$) to low-energy ($r_L \ll \lambda_c^{\parallel}$) particles, transport occurs at rates progressively less than the FLRW rate. An alternative theory ("BAM") based on the Taylor-Green-Kubo (TGK) formulation (Bieber & Matthaeus 1997; Forman 1977) gives a perpendicular diffusion that is weaker than the FLRW at low energy but underestimates the simulation values. Generally speaking, at moderate to low energies, the FLRW and BAM results bracket the numerical results (Giacalone & Jokipii 1999; Mace et al. 2000). A substantial complication is the interaction between perpendicular and parallel scattering, which can reduce perpendicular transport to subdiffusive levels (Lingenfelter, Ramaty, & Fisk 1971; Urch 1977; Rechester & Rosenbluth 1978; Kota & Jokipii 2000; Qin et al. 2002a). Particles parallel-scatter, reversing direction relative to the large-scale guide field. If the field lines sampled by the gyromotion are closely similar to those encountered before the reversal, then perpendicular displacements are suppressed. The property that determines whether perpendicular diffusion is lost (Qin et al. 2002a) or recovered (Qin, Matthaeus, & Bieber 2002b) is evidently the transverse complexity of the magnetic field. Flux surfaces with high traverse complexity (see Fig. 1) are characterized by the rapid separation of nearby field lines. Long ago recognized as important for particle transport (Jokipii 1973), field line separation has been discussed extensively in the context of nonlinear dynamics and fusion devices, in which Coulomb collisions, shear, and other effects (Rechester & Rosenbluth 1978; Kadomtsev & Pogutse 1979) are invoked to explain perpendicular diffusion at non-QLT/FLRW levels.

Some descriptions invoke rapid exponential separation or the "stochastic instability" (Zaslavskii & Chirikov 1972) of field lines and/or weak Coulomb collisions in order to restore diffusion (Rechester & Rosenbluth 1978) in the presence of parallel scattering. Conclusions vary widely. For example, the argument that even vanishingly weak collisions can restore the FLRW level of transport (Rechester & Rosenbluth 1978) is at least partially dependent on the occurrence of exponential stochastic instability up to a scale of the correlation length, a proposition that we find to be questionable in broadband turbulence (see Rax & White 1992). It has also been suggested that the exponential separation itself guarantees the restoration of diffusion (Chandran & Cowley 1998). Counterexamples to these propositions are found in recent numerical results (Qin et al. 2002a, 2002b). For example, perpendicular transport can be suppressed to subdiffusive levels even when there is no ignorable coordinate, although the stochastic instability argument would appear to be in force. Likewise, cases are found in which, lacking collisions entirely, perpendicular diffusion is restored, although to levels lower than the FLRW.

Here we develop a nonlinear theory in which transverse complexity is invoked to decorrelate trajectories after parallel scattering, thus restoring perpendicular diffusive transport. Among our assumptions is that decorrelation of nearby field lines is

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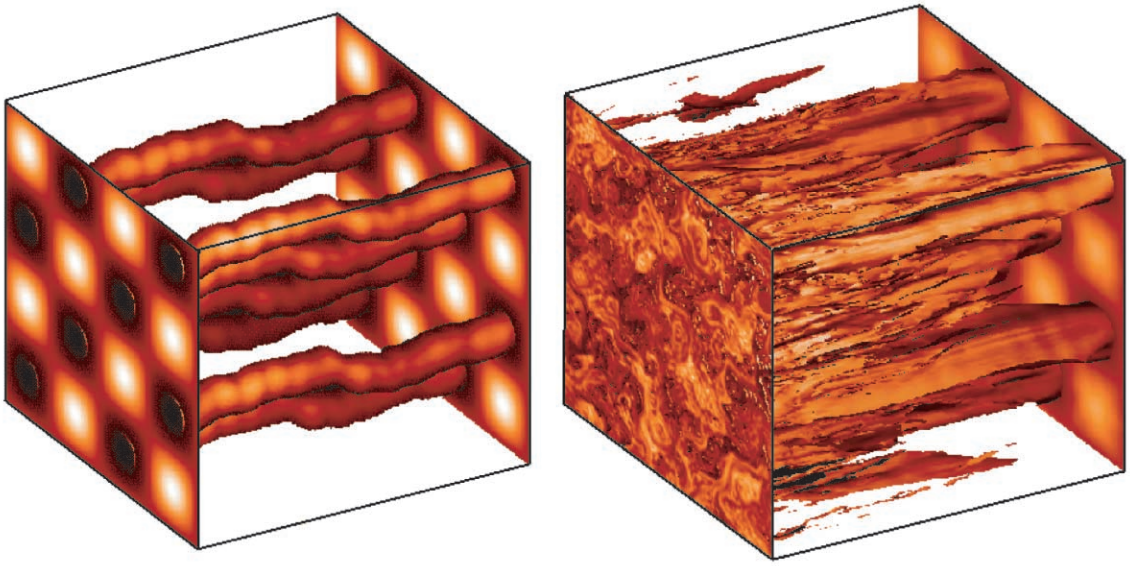

FIG. 1.—Flux surfaces for two magnetic field models. *Left*: Slab model with no transverse structure. The field lines do not separate. *Right*: Two-component model, with 80% of its energy in two-dimensional modes, exhibiting considerable transverse complexity. The nearby field lines rapidly separate.

diffusive, not exponential. We will compare our conclusion and several other theories with direct numerical simulations.

3. FORMULATION OF NONLINEAR PERPENDICULAR DIFFUSION

Consider a statistically homogeneous magnetic field $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$, with a uniform constant $\boldsymbol{B}_0$ oriented in the $\hat{z}$-direction and a random fluctuation $\boldsymbol{b}(\boldsymbol{x}, t)$. Define an ensemble average $\langle \ldots \rangle$ such that $\langle \boldsymbol{B} \rangle = \boldsymbol{B}_0$ and $\langle \boldsymbol{b} \rangle = 0$. The fluctuations are stationary and homogeneous, so the two-point, two-time covariances $R_{ij}(\boldsymbol{r}, \tau) = \langle b_i(\boldsymbol{x}, t) b_j(\boldsymbol{x} + \boldsymbol{r}, t + \tau)\rangle$ vary only with spatial and temporal lags $\boldsymbol{r}$ and τ. For convenience, we specialize to transverse fluctuations $\boldsymbol{b} \cdot \boldsymbol{B}_0 = 0$.

The perpendicular (Fokker-Planck) diffusion coefficient is $\kappa_{xx} = \langle (\Delta x)^2 \rangle / 2\Delta t$, for a component of transverse displacement Δx and a suitable limiting timescale Δt. In numerical work, the running diffusion coefficient $\tilde{\kappa}_{xx} = \frac{1}{2} d\langle (\Delta x)^2 \rangle / dt$ is useful. We will develop an approximate expression for κ_{xx} in the TGK formulation (e.g., Kubo 1957)

$$\kappa_{xx} = \int_0^\infty \langle v_x(0) v_x(t') \rangle dt' \qquad (1)$$

by modeling the integrand, i.e., the two-time single-particle velocity autocorrelation, employing a series of physically motivated approximations.

First, we assume that perpendicular transport is governed by the velocity of gyrocenters that follow field lines. Accordingly, in equation (1), we replace the x-direction velocity v_x with

$$\tilde{v}_x \equiv a v_z b_x / B_0, \qquad (2)$$

where a is a proportionality constant to be determined after the fact. The crucial effect of gyromotion in sampling the transverse structure of the turbulence is not ignored—rather it will be built into additional developments (see below). Using equation (2) converts the TGK integrand into a fourth-order correlation function $\langle v_z(0) b_x[\boldsymbol{x}(0), 0]\, v_z(t') b_x[\boldsymbol{x}(t'), t'] \rangle$ involving particle velocities and magnetic fluctuation values at the particle positions. Next, we assume that the particle velocities are uncorrelated with the local magnetic field vector. This is exact for any distribution symmetric about 90° pitch angle. Thus, the more daunting fourth-order correlation is replaced by a product of second-order correlations, $\langle v_z(0) v_z(t') \rangle \langle b_x[\boldsymbol{x}(0), 0]\, b_x[\boldsymbol{x}(t'), t'] \rangle$. The TGK formula becomes

$$\kappa_{xx} = \frac{a^2}{B_0^2} \int_0^\infty dt' \langle v_z(0) v_z(t') \rangle \langle b_x[\boldsymbol{x}(0), 0] b_x[\boldsymbol{x}(t'), t'] \rangle. \qquad (3)$$

This reduces to the FLRW result in an appropriate limit.[4]

The two-time parallel-velocity autocorrelation is modeled by the (isotropic) assumption $\langle v_z(0) v_z(t') \rangle = (v^2/3) e^{-vt'/\lambda_\parallel}$, where $\lambda_\parallel$ is the mean free path for parallel scattering and v is the particle speed. This is consistent with the TGK definition of $\kappa_{zz} = v\lambda_\parallel / 3 = \int_0^\infty dt' \langle v_z(0) v_z(t') \rangle$.

A key step is the modeling of the Lagrangian magnetic autocorrelation $\langle b_x[\boldsymbol{x}(0), 0]\, b_x[\boldsymbol{x}(t'), t'] \rangle$. QLT replaces the particle trajectory $\boldsymbol{x}(t')$ by an unperturbed trajectory. Here we treat $\boldsymbol{x}(t')$ as a random variable and employ Corrsin's independence hypothesis (Corrsin 1959; Salu & Montgomery 1977; McComb 1990), thus retaining the stochastic character of the particle position vector. In real space form,

$$\langle b_x[\boldsymbol{x}(0), 0] b_x[\boldsymbol{x}(t'), t'] \rangle = \int R_{xx}(\boldsymbol{y}, t') P(\boldsymbol{y}|t') d^3y, \qquad (4)$$

where $R_{xx}(\boldsymbol{y}, t')$ is the Eulerian two-point, two-time correlation and $P(\boldsymbol{y}|t')$ is the probability density of the particle having displacement $\boldsymbol{y}$ at time t'. We find that

$$\kappa_{xx} = \frac{a^2 v^2}{3} \int d^3k \left[\frac{S_{xx}(\boldsymbol{k})}{B_0^2} \int_0^\infty dt'\, e^{-vt'/\lambda_\parallel} \Gamma_{xx}(\boldsymbol{k}, t') \langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}(t')} \rangle \right], \qquad (5)$$

where we introduce the spectral amplitude $S_{xx}(\boldsymbol{k}, t) =$

[4] For constant v_z, $a = 1$, and a one-dimensional slab model magnetic field, eq. (3) becomes $\kappa_{xx} = a^2 v_z^2 / B_0^2 \int_0^\infty \langle b_x(0) b_x(z = v_z t') \rangle dt' \simeq v D_\perp / 2$, where $D_\perp$ is the Fokker-Planck coefficient for field line wandering (Jokipii 1966).

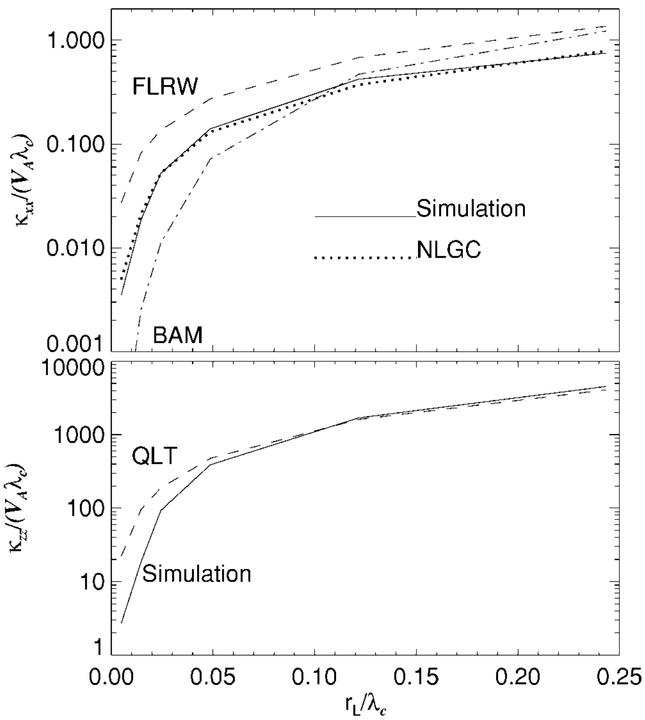

FIG. 2.—Perpendicular (*upper panel*) and parallel (*lower panel*) diffusion coefficient as a function of r_L/λ_c, with $b/B_0 = 0.2$ and $E^{\rm slab} : E^{\rm 2D} = 20 : 80$. The particle velocity varies by a factor of 50. *Upper panel, solid line*: κ_{xx} from numerical simulation; *dotted line*: κ_{xx} from present NLGC theory (eq. [7]); *dashed line*: κ_{xx} from FLRW limit; *dash-dotted line*: κ_{xx} from BAM theory. *Lower panel, solid line*: κ_{zz} from numerical simulation; *dashed line*: κ_{zz} from QLT.

$(2\pi)^{-3}\int_{-\infty}^{\infty} R_{xx}(\boldsymbol{\zeta}, t')e^{-i\boldsymbol{k}\cdot\boldsymbol{\zeta}}\,d^3\zeta$; for convenience, we define $\Gamma_{xx}(\boldsymbol{k}, t')$ so that $S_{xx}(\boldsymbol{k}, t') = S_{xx}(\boldsymbol{k})\Gamma_{xx}(\boldsymbol{k}, t')$.

Proceeding, we assume that the components of the trajectory have uncorrelated axisymmetric Gaussian distributions $P(\boldsymbol{y}|t')$ and, furthermore, that the distribution of displacements is diffusive for all values of time. This immediately leads to

$$\langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}(t')}\rangle = e^{-k_\perp^2\kappa_{xx}t' - k_\parallel^2\kappa_{zz}t'}, \tag{6}$$

where $k_\perp^2 = k_x^2 + k_y^2$ and $k_\parallel = k_z$, thus achieving a statistical closure for κ_{xx} in terms of κ_{zz} and κ_{xx} itself. The above assumption stands in formal contrast to the assumption that trajectories (or field lines) separate exponentially (Rechester & Rosenbluth 1978; Chandran & Cowley 1998).

Using these approximations and for simplicity letting $\Gamma_{xx}(\boldsymbol{k}, t') = e^{-\gamma(\boldsymbol{k})t'}$, equation (5) becomes, after an elementary integration,

$$\kappa_{xx} = \frac{a^2v^2}{3B_0^2}\int \frac{S_{xx}(\boldsymbol{k})dk_x\,dk_y\,dk_\parallel}{\frac{v}{\lambda_\parallel} + (k_x^2 + k_y^2)\kappa_{xx} + k_\parallel^2\kappa_{zz} + \gamma(\boldsymbol{k})}. \tag{7}$$

This equation determines κ_{xx} subject to our understanding of the parallel mean free path, the dynamical decay rate, and the form of the spectrum. We refer to it as the nonlinear guiding center (NLGC) theory.

4. NUMERICAL SIMULATION RESULTS

Numerical tests are employed to assess the accuracy of the NLGC theory, for the particular case of a static $[\gamma(\boldsymbol{k}) = 0]$ two-component, slab/two-dimensional model magnetic field[5] with a spectral index $\nu = \frac{5}{6}$. The spectral bend-over scales λ and λ_x, in the z- and x-direction, respectively (proportional to the respective correlation lengths λ_c and $\lambda_{c,x}$), are chosen so that $\lambda = 10\lambda_x$ (and therefore $\lambda_{c,x} \ll \lambda_c$). This produces a strong transverse complexity that guarantees diffusive perpendicular transport. We fix the ratio of the energies $E^{\rm slab} \equiv \langle b_{\rm slab}^2\rangle$ and $E^{\rm 2D} \equiv \langle b_{\rm 2D}^2\rangle$ in the slab and two-dimensional components,[6] respectively, while controlling the ratio of the rms fluctuation amplitude to DC magnetic field strength b/B_0. We choose the numerical factor to be $a = 1/\sqrt{3}$.

The diffusion coefficient is computed numerically[7] using the procedures described previously (Giacalone & Jokipii 1999; Qin et al. 2002a). The ratio of the gyroradius to the parallel correlation scale, r_L/λ_c, is varied by changing the particle speed. For the selected parameters, using a single realization of the turbulence, 1000 particle trajectories are computed over timescales vt/λ approximately equal to several hundred to 1000, and the running diffusion coefficients $\tilde{\kappa}_{xx}$ and $\tilde{\kappa}_{zz}$ are computed. A usable value of κ_{xx} is obtained when a stable "flat" regime is observed for a period of several hundred vt/λ. In all cases in which a stable regime of $\tilde{\kappa}_{xx}$ is observed, $\tilde{\kappa}_{zz}$ has already attained a stable value. Thus, the diffusion coefficients we report here are in the regime of the "second" diffusion (Qin et al. 2002b).

Figure 2 shows the results for relatively weak turbulence, $b/B_0 = 0.2$, and the spectral components in the ratio $E^{\rm slab} : E^{\rm 2D} = 20 : 80$, believed to be consistent with solar wind observations (Bieber et al. 1996). The ratio r_L/λ_c is varied from 1/200 to 1/4, and the results are compared with the present theory and the FLRW and BAM predictions. The simulations show that κ_{zz} is comparable to, or somewhat less than, the QLT result. FLRW and BAM results bracket the numerically determined κ_{xx} at low energies as previously reported (Giacalone & Jokipii 1999; Mace et al. 2000) . The present theory provides much better agreement with the computed values.

Another set of numerical results is compared with theory in Figure 3, for parameters similar to those above, but with stronger turbulence, $b/B_0 = 1.0$. Here we show an additional comparison, with a theoretical result designated as "CC&RR," $\kappa_{xx} = (vD_\perp/3)\lambda_\parallel/[\lambda_c \log(\lambda_c/r_L)]$. This is essentially a collisionless adaptation of the Rechester & Rosenbluth (1978) and Stix (1978) result for perpendicular transport in weakly collisional toroidal plasmas, as formulated by Chandran & Cowley (1998). In contrast to the present theory, CC&RR assumes an exponential separation of field lines to at least the correlation scale. Once again, we see that for nearly 2 orders of magnitude in r_L/λ_c, the present theory provides the best account of the computed values.

5. SUMMARY AND DISCUSSION

The above sections outline a theory of the perpendicular diffusion of charged particles, based on the hypothesis that

[5] The two-component, slab/two-dimensional spectrum model (Bieber, Wanner, & Matthaeus 1996) ignores the usually smaller parallel variance and includes only excitations with wavevectors either purely parallel to or purely perpendicular to the mean magnetic field $\boldsymbol{B}_0$, leading to $S_{xx}(\boldsymbol{k}) = S_{xx}^{\rm 2D}(k_\perp)\delta(k_z) + S_{xx}^{\rm slab}(k_z)\delta(\boldsymbol{k}_\perp)$, where, e.g., we choose $S_{xx}^{\rm slab}(k_z) = C(\nu)\lambda\langle b_{\rm slab}^2\rangle(1 + k_z^2\lambda^2)^{-\nu}$, and $S_{xx}^{\rm 2D}(k_\perp) = C(\nu)\lambda_x\langle b_{\rm 2D}^2\rangle(1 + k_\perp^2\lambda_x^2)^{-\nu}/\pi k_\perp$, where $C(\nu) = (2\pi^{1/2})^{-1}\Gamma(\nu)/\Gamma(\nu - \frac{1}{2})$.

[6] Typically, the slab component is generated by choosing 2^{22} Fourier modes with a fixed spectral shape, and random phases, in a large periodic box of size $10^4\lambda$. The two-dimensional component is generated similarly in a 4096 × 4096 Fourier series in a box of size $10^2\lambda$ (see Qin et al. 2002b).

[7] Particles are integrated using a fourth-order adaptive-step Runge-Kutta method with a relative error control set to 1 part per billion.

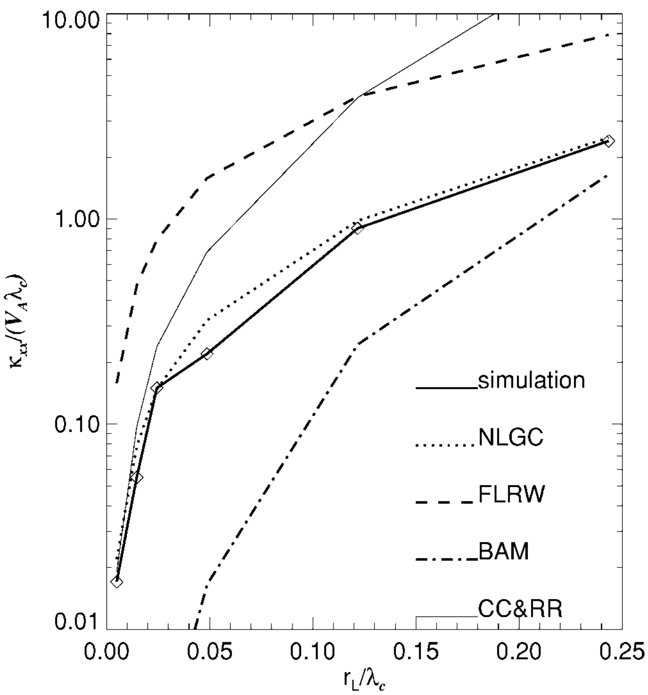

FIG. 3.—Perpendicular diffusion coefficients as a function of r_L/λ_c, with $b/B_0 = 1$ and $E^{slab} : E^{2D} = 20 : 80$. *Thick solid line*: κ_{xx} from numerical simulation; *dotted line*: κ_{xx} from NLGC theory (eq. [7]); *thin solid line*: CC&RR theory; *dashed line*: FLRW limit; *dash-dotted line*: BAM theory. The turbulence amplitude is larger than in Fig. 2, and parallel diffusion (not shown) is no longer accurately given by QLT for these parameters. The NLGC theory is more accurate than the other theories shown.

decorrelation, which is necessary to achieve a diffusive random walk, is accomplished by particle gyrocenters following magnetic field lines, which themselves diffusively separate as a result of the transverse complexity of the turbulence. The effect of parallel (pitch angle) scattering as well as the influence of dynamical turbulence are included explicitly through the diffusion assumption, which, along with Corrsin's independence hypothesis, allows a reasonably general formulation of perpendicular diffusion. At no point have weak Coulomb collisions or the exponential separation of field lines been invoked. Through comparison with direct numerical simulations of test particles, we have shown evidence that this NLGC theory of perpendicular diffusion behaves well for a wide range of test particle parameters and is noticeably better than several other theories of perpendicular transport included for comparison. The present theory also predicts that the ratio $\kappa_{xx}/\kappa_{zz} \approx 0.02$–0.05 for the range of parameters considered, consistent with previous reports (Giacalone & Jokipii 1999). We are currently studying the application of the present theory in heliospheric and cosmic-ray physics in the hope that an improved formulation of perpendicular diffusion might be useful in solving a number of observational puzzles.

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